

The Yagita invariant of general linear groups

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We give a definition of the Yagita invariant $p^\circ(G)$ of an arbitrary group G , and compute $p^\circ(GL_n(\mathcal{O}))$ for each prime p , where \mathcal{O} is any integrally closed subring of the complex numbers \mathbf{C} (i.e., \mathcal{O} is integrally closed in its field of fractions F , which is also a subring of \mathbf{C}). We also show that $p^\circ(SL_n(\mathcal{O}))$ is equal to $p^\circ(GL_n(\mathcal{O}))$ for $n \geq 2$ except that possibly $p^\circ(SL_n(\mathcal{O}))$ is $1/2p^\circ(GL_n(\mathcal{O}))$ for some small n and ‘small’ rings \mathcal{O} . Our definition of $p^\circ(G)$ extends both Yagita’s original definition for G finite [9] and the definition given by one of us for G of finite virtual cohomological dimension (or vcd) [8]. For G of finite vcd such that the Tate-Farrell cohomology $\hat{H}^*(G)$ of G is p -periodic, $p^\circ(G)$ is equal to the p -period. Hence our results may be viewed as a generalisation of those of B urgisser and Eckmann [1,2], who compute the p -periods of $GL_n(\mathcal{O})$ and $SL_n(\mathcal{O})$ for various \mathcal{O} such that these groups have finite vcd and for all n such that these groups are p -periodic. The methods we use are similar to those used in [1].

Notation. Throughout this paper p shall be a fixed prime number, \mathcal{O} an integrally closed subring of \mathbf{C} , and F the field of fractions of \mathcal{O} . ζ_p shall be a primitive p th root of unity in \mathbf{C} , and l shall be the degree of $F[\zeta_p]$ as an extension of F . In [1,2,4], the notation $\phi_F(p)$ is used for l .

Definition. Let G be a group. If C is an order p subgroup of G , then

$$H^*(BC; \mathbf{Z}) = \mathbf{Z}[x]/(px)$$

for x a generator of H^2 . Define $n(C)$ (either a positive integer or infinity) to be the supremum of the integers n such that the image of $H^*(BG)$ in $H^*(BC)$ is contained in the subalgebra generated by x^n . Now define the Yagita invariant $p^\circ(G)$ to be

$$p^\circ(G) = \text{l.c.m.}\{2n(C) : C \leq G, \quad |C| = p\},$$

where the least common multiple of the empty set is equal to 1, and the least common multiple of an unbounded set of integers, or of a set containing infinity, is infinity. The invariant $p^\circ(G)$ depends on the prime p as well as on the group G , but we have resisted the temptation to write, for example, $2^\circ(G)$ for $p^\circ(G)$ in the case when $p = 2$.

The following result is immediate from the definition.

Proposition 1. *Let $f: H \rightarrow G$ be a group homomorphism whose kernel contains no element of order p . (For example, f might be injective, or the kernel of f might be torsion-free.) Then $p^\circ(H)$ divides $p^\circ(G)$. \square*

It is easy to see that $p^\circ(G)$ is finite if G is finite, and Proposition 1 applied in the case when G is the quotient of H by a torsion-free normal subgroup of

finite index implies that $p^\circ(H)$ is finite if H is a group of finite vcd (see [5] for a more detailed proof of this fact). Using Chern classes, we shall show in Corollary 7 that if G admits a faithful finite-dimensional representation over \mathbf{C} , then $p^\circ(G)$ is finite.

Some remarks concerning the origin of the definition of $p^\circ(G)$ and its geometric motivation are in order. N. Yagita made the above definition for finite groups [9], and proved that if a finite group G acts freely on a (finite) product $S^{m-1} \times \dots \times S^{m-1}$ of spheres of the same dimension $m-1$, with trivial action on their homology, then $p^\circ(G)$ divides m . This result may be viewed as a generalisation of the theorem bounding the dimension of a sphere with a free G -action in the case when the cohomology of G is periodic. One of us suggested extending the definition to groups of finite vcd in [8]. The connection with actions on products of spheres extends to this case too. If G is a group of finite vcd acting freely, properly discontinuously and with trivial action on homology, on a finite product $S^{m-1} \times \dots \times S^{m-1} \times \mathbf{R}^n$, then $p^\circ(G)$ divides m , by the same argument as used in [9] for the case when G is finite. For arbitrary groups, the connection with the geometry of group actions is less clear. One reason for extending the definition to arbitrary groups is that this makes our results easier to state and no harder to prove.

To motivate our work we start by stating the main theorems and some corollaries. The proofs of the main theorems will occupy the remainder of the paper.

Theorem 2. *As explained under the title ‘Notation’, let \mathcal{O} be an integrally closed subring of \mathbf{C} with field of fractions F , and let $l = |F[\zeta_p] : F|$. Define $\psi(t)$, for t a positive real, to be the greatest integer power of p less than or equal to t . The Yagita invariant $p^\circ(GL_n(\mathcal{O}))$ is given by the following table.*

$$p^\circ(GL_n(\mathcal{O})) = \begin{cases} 1 & \text{for } n < l \\ 2\text{l.c.m.}\{m : l|m, \quad m|(p-1), \quad m \leq n\} & \text{for } l \leq n \leq p-1 \\ 2(p-1)\psi(n/l) & \text{for } n \geq p \end{cases}$$

Remark. The above theorem holds for all primes p , but the statement can be simplified slightly for $p = 2$. In this case $l = 1$, and $p^\circ(GL_n(\mathcal{O})) = 2\psi(n)$ for all $n \geq 1$. Note that $p^\circ(GL_n(\mathcal{O}))$ depends only on F , so that if R is any subring of \mathbf{C} such that $\mathcal{O} \leq R \leq F$, then $p^\circ(GL_n(R)) = p^\circ(GL_n(\mathcal{O}))$.

Replacing $GL_n(\mathcal{O})$ by $SL_n(\mathcal{O})$ does not change the Yagita invariant, except for small values of n and ‘small’ rings \mathcal{O} , when it may be necessary to introduce a factor of $1/2$. Thus one obtains the following.

Theorem 3. *With notation as above, for $n \geq 2$, $p^\circ(SL_n(\mathcal{O}))$ is equal to $p^\circ(GL_n(\mathcal{O}))$ except that $p^\circ(SL_n(\mathcal{O}))$ may be equal to $\frac{1}{2}p^\circ(GL_n(\mathcal{O}))$ in the following cases.*

- a) $p = 2$, $n = 2$, and \mathcal{O} contains no square root of -1 .

b) p is odd, n has the form $n = 2^r l$ for 2^r dividing $(p-1)/l$, and \mathcal{O} contains no n th root of -1 .

Corollary 4. Define the function ψ as in the statement of Theorem 2. Then for $n \geq 2$ and any prime p , the Yagita invariant of $GL_n(\mathbf{Z})$ is as follows.

$$p^\circ(GL_n(\mathbf{Z})) = \begin{cases} 1 & \text{for } n < p-1, \\ 2(p-1)\psi(n/(p-1)) & \text{for } n \geq p-1. \end{cases}$$

For all $n \geq 2$ and all p , $p^\circ(SL_n(\mathbf{Z})) = p^\circ(GL_n(\mathbf{Z}))$, except that for p odd, $p^\circ(SL_{p-1}(\mathbf{Z})) = p-1$, and for $p=2$, $p^\circ(SL_2(\mathbf{Z})) = 2$.

Proof. The statement for the general linear groups is a special case of Theorem 2. The only cases when $p^\circ(SL_n(\mathbf{Z}))$ is not completely determined by Theorem 3 are those when $n = p-1$ and when $n = p = 2$, but in these cases $SL_n(\mathbf{Z})$ is p -periodic and the p -period is determined in [1]. \square

The following statement may be proved in the same way as Theorems 2 and 3, and could be considered as a corollary but for the weaker conditions imposed on the subring R of \mathbf{C} .

Theorem 5. Let R be any subring of \mathbf{C} containing ζ_p , a primitive p th root of 1, and let ψ be as in the statement of Theorem 2. Then the Yagita invariant of $GL_n(R)$ is given by the following table.

$$p^\circ(GL_n(R)) = \begin{cases} 2\text{l.c.m.}\{m : m|(p-1), \quad m \leq n\} & \text{for } n \leq p-1 \\ 2(p-1)\psi(n) & \text{for } n \geq p \end{cases}$$

For R as above, $p^\circ(SL_n(R)) = p^\circ(GL_n(R))$ if any of the following conditions is satisfied:

- a) $n \geq \max\{p, 3\}$,
- b) $n \geq 2$, p is odd, and R contains a $(p-1)$ st root of -1 ,
- c) $n = 2$, $p = 2$, and R contains a square root of -1 .

Our upper bound for $p^\circ(GL_n(\mathcal{O}))$ uses the Chern classes of the natural representation in $GL_n(\mathbf{C})$, and relies on the following proposition.

Proposition 6. Let $f(X)$ be a polynomial over the field \mathbf{F}_p , all of whose roots lie in \mathbf{F}_p^\times . If there is a polynomial g and integer n such that $f(X) = g(X^n)$, then n has the form mp^q for some m dividing $p-1$ and some positive integer q .

Proof. Let $n = mp^q$, where p does not divide m . It remains to prove that m divides $p-1$. Now $g(X^n) = g(X^m)^{p^q}$, so without loss of generality we may assume that $q = 0$. Now if $g(Y) = 0$ has roots y_1, \dots, y_k , then the roots of $g(X^m) = 0$ are the roots of $y_i - X^m = 0$ for each i , and because p does not divide m these polynomials have no repeated roots. If y_i is not an element of \mathbf{F}_p^\times then $y_i - X^m$ can have no roots in \mathbf{F}_p . It has exactly m roots in \mathbf{F}_p if and only if the inverse image of y_i under the map $x \mapsto x^m$ from \mathbf{F}_p^\times to itself has order m , which can only happen if m divides $p-1$. \square

We will use the following Corollary in the proof of Theorem 2.

Corollary 7. *With notation as in the statement of Theorem 2, let G be a subgroup of $GL_n(F)$. Then the Yagita invariant $p^\circ(G)$ divides the number given for $p^\circ(GL_n(\mathcal{O}))$ in Theorem 2.*

Proof. For each order p subgroup C of G , we bound the number $n(C)$ occurring in the definition of $p^\circ(G)$ by considering the image of the cohomology $H^*(BGL_n(\mathbf{C}))$ of the Lie group $GL_n(\mathbf{C})$ in $H^*(BC)$ under the inclusion of C in $GL_n(\mathbf{C})$. Recall (or see for example [7]) that $H^*(BGL_n(\mathbf{C}))$ is a free polynomial algebra on generators c_1, \dots, c_n , the *universal Chern classes*, where c_i has degree $2i$. For a representation ρ of a group H in $GL_n(\mathbf{C})$, the element $\rho^*(c_i)$ is usually written $c_i(\rho)$, and called the i th Chern class of the representation ρ . The sum $c.(\rho)$ of all the $c_i(\rho)$ is known as the total Chern class of ρ , and has the property that $c.(\rho \oplus \rho') = c.(\rho)c.(\rho')$.

If $H^*(BC) = \mathbf{Z}[x]/(px)$, then the total Chern classes of the p distinct 1-dimensional complex representations of C are $1 + ix$ for $i \in \mathbf{F}_p$, where the case $i = 0$ corresponds to the trivial representation. If we fix a generator of C such that the 1-dimensional representation of C sending this generator to ζ_p has total Chern class $1 + x$, then the representation sending this same generator to ζ_p^i has total Chern class $1 + ix$. For F as in the statement, there are $1 + (p-1)/l$ irreducible F -representations of C , the trivial representation of dimension one and $(p-1)/l$ others of dimension l . Over \mathbf{C} the faithful representations split as a direct sum of one copy each of the representations sending a fixed generator of C to $\zeta_p^{ri^j}$, for some r , where i generates a subgroup of \mathbf{F}_p^\times of order l and $0 \leq j < l$. It follows that the total Chern classes of these representations have the form $1 - ix^l$, where i ranges over the $(p-1)/l$ distinct l th roots of unity in \mathbf{F}_p .

Now the ring $H^*(BC) \otimes \mathbf{F}_p$ is isomorphic to the free polynomial ring $\mathbf{F}_p[x]$, and the total Chern class of the inclusion of C in $GL_n(\mathbf{C})$ is a polynomial $f(x)$ of degree less than or equal to n satisfying the hypotheses of Proposition 6. Moreover, since this inclusion factors through $GL_n(F)$, the total Chern class can be viewed as a polynomial $\tilde{f}(y)$ in $y = x^l$ of degree less than or equal to n/l satisfying the hypotheses of the Proposition. This polynomial is not the trivial polynomial because the inclusion of C is not the trivial representation. The integer $n(C)$ (resp. $n(C)/l$) is the greatest integer m such that $f(x)$ (resp. $\tilde{f}(y)$) can be expressed as a polynomial in x^m (resp. y^m), and so by Proposition 2 each of $n(C)$ and $n(C)/l$ is a divisor of $p-1$ multiplied by a power of p . Note also that $n(C)$ is at most n . This implies that for each C , the number $n(C)$ divides the bound given in Theorem 2, and hence so does the l.c.m. of all such $n(C)$. \square

Remark. The discussion of the total Chern classes of F -representations of the cyclic group of order p could be avoided by quoting the general bounds on the orders of Chern classes of F -representations given in [4]. Corollary 7 also holds for groups G having a representation in $GL_n(F)$ which is faithful on

every order p subgroup, by the same proof. Corollary 7 may be compared with Lemma 1.2 of [5].

Our lower bounds for $p^\circ(GL_n(\mathcal{O}))$ and $p^\circ(SL_n(\mathcal{O}))$ use various finite subgroups, which we define below.

Definition. For p an odd prime, and m a divisor of $p-1$, let $G_1(p, m)$ be the split metacyclic group $C_p : C_m$, where C_m acts faithfully on C_p , and let $G_2(p, m)$ be the split metacyclic group $C_p : C_{2m}$, where C_{2m} acts via the faithful action of its quotient C_m . For p an odd prime let $E(p, 1)$ be the non-abelian group of order p^3 and exponent p , and let $E(2, 1)$ be the dihedral group of order eight. For any p , and any positive integer m , let $E(p, m)$ be the central product of m copies of $E(p, 1)$, so that $E(p, m)$ is one of the two extraspecial groups of order p^{2m+1} . In ATLAS notation [3], the group $E(p, m)$ is called p_+^{1+2m} . Since the symbol p is already overworked in this paper we shall not use the ATLAS notation here.

Lemma 8. *For p an odd prime the Yagita invariants of the groups defined above are as follows.*

$$p^\circ(G_1(p, m)) = p^\circ(G_2(p, m)) = 2m, \quad p^\circ(E(p, m)) = 2p^m$$

For $p = 2$, let Q_8 be the quaternion group of order 8. Then

$$p^\circ(E(2, m)) = 2^{m+1} \quad \text{and} \quad p^\circ(Q_8) = 4.$$

Proof. The groups $G_1(p, m)$ and $G_2(p, m)$ are p -periodic, and their p -periods are shown to be as claimed in Proposition 3.1 and Section 3.3 of [2]. The Yagita invariant for the extraspecial groups was computed in Example B of Section 2 of [9], which in turn used results of [6]. \square

Lemma 9. *For p an odd prime, and m a divisor of $p-1$, let $n = ml/(m, l)$. The group $G_1(p, m)$ embeds in $GL_n(\mathcal{O})$, and in $SL_{n+1}(\mathcal{O})$. If either $l/(m, l)$ is even or m is odd (and greater than 1) then $G_1(p, m)$ embeds in $SL_n(\mathcal{O})$. If m is even and \mathcal{O} contains an m th root of -1 , then $G_2(p, m)$ embeds in $SL_n(\mathcal{O})$. The group $E(p, m)$ embeds in $SL_{lp^m}(\mathcal{O})$.*

For $p = 2$ and $m > 1$ the group $E(2, m)$ embeds in $SL_{2^m}(\mathbf{Z})$. The group $E(2, 1)$ (the dihedral group of order 8) embeds in $GL_2(\mathbf{Z})$ and in $SL_3(\mathbf{Z})$. The quaternion group of order eight embeds in $SL_2(\mathbf{Z}[i])$.

Proof. First consider the case when p is odd. For $G_1(p, m)$ and $G_2(p, m)$ this is essentially contained in [2] Section 3.2. If m divides m' , then $G_1(p, m)$ is a subgroup of $G_1(p, m')$, so without loss of generality we may assume that m is divisible by l and show that in this case $G_1(p, m)$ embeds in $GL_m(\mathcal{O})$. As an \mathcal{O} -module, $\mathcal{O}[\zeta_p]$ is free of rank l , which is closed under multiplication by ζ_p and under the action of $\text{Gal}(F[\zeta_p]/F)$, which together generate a group of \mathcal{O} -linear

automorphisms of $\mathcal{O}[\zeta_p]$ isomorphic to $G_1(p, l)$. If m is a proper multiple of l , view $G(p, l)$ as a subgroup of $G(p, m)$, and the induced representation V coming from the above representation is an m -dimensional faithful representation of $G(p, m)$ over \mathcal{O} .

It is easy to check that the determinant of the action of an element of order p of $G_1(p, m)$ on V is equal to 1, and that $V \otimes F$ restricts to a cyclic subgroup of $G_1(p, m)$ of order m as a sum of $l/(m, l)$ copies of the regular representation (use the normal basis theorem). Thus if either m is odd or $l/(m, l)$ is even, the determinant of an element of order m acting on V is 1, and so $G_1(p, m)$ is contained in $SL(V) \cong SL_n(\mathcal{O})$. In any case, the determinant of the action of $G_1(p, m)$ on V has image contained in $\{\pm 1\} \subseteq \mathcal{O}^\times$, so the action of $G_1(p, m)$ on $V \oplus \Lambda^n(V)$ gives an embedding from $G_1(p, m)$ into $SL_{n+1}(\mathcal{O})$.

If m is even and $l/(m, l)$ is odd, then \mathcal{O} contains an m th root of -1 if and only if \mathcal{O} contains an n th root of -1 . In this case, if μ is an n th root of -1 in \mathcal{O} , and A of order p and B of order m with $\det(B) = -1$ generate a subgroup of $GL_n(\mathcal{O})$ isomorphic to $G_1(p, m)$, then A and μB generate a subgroup of $SL_n(\mathcal{O})$ isomorphic to $G_2(p, m)$.

Now consider the extraspecial group $E(p, m)$ for p odd. Note that the centre Z of $E(p, m)$ is cyclic of order p , and that $E(p, m)$ has a subgroup of index p^m containing Z as a direct factor. This subgroup has an l -dimensional \mathcal{O} -representation which is faithful on Z , and the corresponding induced $E(p, m)$ -module is an lp^m -dimensional representation which must be faithful (since any nontrivial normal subgroup of a p -subgroup meets the centre nontrivially). Over \mathbf{C} , $E(p, m)$ has $(p-1)$ faithful irreducible representations, each of dimension p^m (arising as induced modules in the above way for different choices of 1-dimensional modules for Z). Using characters it is easy to see that these representations restrict to any non-central subgroup of order p as a sum of copies of the regular representation and to the centre as a sum of copies of a single irreducible representation, and hence that they have image in $SL_{p^m}(\mathbf{C})$ (note that $E(p, m)$ has exponent p). The representations over \mathcal{O} constructed above split over \mathbf{C} into a sum of some of the faithful irreducibles, so have determinant 1.

For $p = 2$, there is a unique faithful irreducible complex representation of $E(2, m)$, which has dimension 2^m , and an argument similar to the above shows that it is realisable over \mathbf{Z} . Using characters one can show that the restriction of this representation to any cyclic subgroup of order four lies in $SL_{2^m}(\mathbf{Z})$, and that the restriction to any non-central subgroup of order two is isomorphic to a sum of 2^{m-1} copies of the regular representation, so lies in $SL_{2^m}(\mathbf{Z})$ provided that $m \geq 2$.

The left action of the quaternion group Q_8 on $\mathbf{Z}[i, j, k]$ commutes with the right action of $\mathbf{Z}[i]$, giving a faithful representation of Q_8 which has image in $SL_2(\mathbf{Z}[i])$. \square

Proof of Theorem 2. Corollary 7 gives an upper bound for $p^\circ(GL_n(\mathcal{O}))$. For $n \leq p-1$, for each $m \leq n$ such that l divides m and m divides $p-1$,

Lemma 9 tells us that $G_1(p, m)$ occurs as a subgroup of $GL_n(\mathcal{O})$, and so by Proposition 1 and Lemma 8, $p^\circ(GL_n(\mathcal{O}))$ is divisible by $p^\circ(G_1(m, p)) = 2m$. This gives the bound for $n \leq p - 1$, and shows that for $n \geq p$, $p^\circ(GL_n(\mathcal{O}))$ is divisible by $2(p - 1)$. Now (for any p) the group $E(p, m)$ is a subgroup of $GL_n(\mathcal{O})$ for each $n \geq 2lp^m$, and has Yagita invariant $2p^m$. This gives the p -part of the bound for $n \geq p$. \square

Proof of Theorem 3. This is similar to the proof of Theorem 2. In the case when $n \leq p - 1$, the l.c.m. occuring in the expression given for $p^\circ(GL_n(\mathcal{O}))$ is clearly equal to the following expression.

$$\text{l.c.m.}\{m : m \leq n, m = lq^r \text{ for some prime } q, q^r | (p - 1)/l\}$$

In other words, we need only consider those m of the form lq^r for some prime q such that q^r divides $(p - 1)/l$. If q is an odd prime and q divides l exactly s times, then $G_1(p, q^{r+s})$ is a subgroup of $SL_n(\mathcal{O})$ for $n = lq^r$, and has Yagita invariant $2q^{r+s}$. If 2 divides l exactly s times, and $2^r l$ divides $p - 1$, then $G_1(p, 2^{r+s})$ is a subgroup of $SL_{n+1}(\mathcal{O})$ for $n = l2^r$ and has Yagita invariant 2^{1+r+s} . From these examples it already follows that $p^\circ(SL_n(\mathcal{O}))$ is divisible by $2(p - 1)$ for $n \geq p$, and that for $n \leq p - 1$, $p^\circ(SL_n(\mathcal{O}))$ is equal to $p^\circ(GL_n(\mathcal{O}))$ except possibly if n is of the form $2^r l$ and is a factor of $p - 1$, when the Yagita invariant for $SL_n(\mathcal{O})$ might be half the Yagita invariant for $GL_n(\mathcal{O})$. If \mathcal{O} contains a 2^{r+s} th root of -1 , or equivalently an n th root of -1 (where $n = l2^r$), then $G_2(p, 2^{r+s})$ is a subgroup of $SL_n(\mathcal{O})$, and has Yagita invariant 2^{1+r+s} , so that in this case too $p^\circ(SL_n(\mathcal{O})) = p^\circ(GL_n(\mathcal{O}))$. \square

Proof of Theorem 5. The groups $GL_n(R)$ and $SL_n(R)$ are subgroups of $GL_n(\mathbf{C})$, so their Yagita invariants are bounded above by $p^\circ(GL_n(\mathbf{C}))$. By the hypothesis on R , the cyclic group C_p admits a faithful representation in $GL_1(R)$. As in Lemma 9 one may use induced representations of the groups $G_i(p, m)$ and $E(p, m)$ to give lower bounds equal to the above upper bounds. We leave the details as an exercise. \square

Remark. The methods that we use also gives some information concerning the Yagita invariant of the groups $G(\mathcal{O})$ for other algebraic groups G . We hope to address this question in a future publication.

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